Monoidal intervals on three- and four-element sets

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Abstract. The aim of this paper is to give an overview about monoidal intervals on three- and four-element sets. Furthermore, two uncountable monoidal intervals on three-element sets are presented in the paper, and we describe some infinite families of collapsing monoids.

1. Introduction

The importance of monoidal intervals comes from the fact that they are closely related to one of the central themes in universal algebra: composition of operations. Sets of operations that are closed under composition naturally arise in many areas of mathematics. P. Hall [11] was lead to the concept of a clone, which can be defined as a composition-closed set of operations containing all projection operations, by studying the word problem for various classes of groups. For an arbitrary set A, the set of all clones on A constitutes a complete lattice with respect to set-theoretic inclusion, this lattice will be denoted by $\mathcal{L}_A$.

E. L. Post started to investigate composition-closed sets of truth functions (that is, composition-closed sets of operations on a 2-element set) in order to understand all possible propositional calculi in 2-valued logic. Post's result in [25] gives a complete description of all members of the clone lattice $\mathcal{L}_{\{0,1\}}$. It turns out that $\mathcal{L}_{\{0,1\}}$ has cardinality $\aleph_0$.

However, the situation changes dramatically when A has more than two elements. In [12] Yu. I. Janov and A. A. Mučnik proved that on a finite set A with more than two elements there are $2^{\aleph_0}$ clones, and the structure of the clone lattice on A is rather complicated. A. A. Bulatov in [1] proved that if $|A| \geqslant 4$ then any direct product of countably many finite lattices can be embedded into the clone lattice $\mathcal{L}_A$.

Next we explain how the study of monoidal intervals may help to understand the structure of the clone lattice better.

Let A be a set. For arbitrary clone $\mathcal{C}$ on A the set of all unary operations in $\mathcal{C}$ is clearly a transformation monoid on A. Furthermore, it is not hard to show (see Á. Szendrei [31], Proposition 3.1) that for arbitrary transformation monoid

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M on A the clones in which the set of unary operations coincides with M form an interval $\mathrm{Int}(M)$ in the clone lattice $\mathcal{L}_A$. Such an interval is called a monoidal interval. If A is finite, then there are only finitely many transformation monoids on A. Hence the monoidal intervals $\mathrm{Int}(M)$ partition the clone lattice $\mathcal{L}_A$ into finitely many blocks. Since $\mathcal{L}_A$ has cardinality $2^{\aleph_0}$ if $|A| \geqslant 3$, one might expect that 'for most transformation monoids M' the corresponding monoidal intervals contain uncountably many clones. This expectation is justified by the fact that if $|A| = 3$, then more than half of the monoidal intervals have cardinality $2^{\aleph_0}$. Nevertheless, it turns out that for many interesting transformation monoids M the interval $\mathrm{Int}(M)$ is countable. So, studying these intervals may lead to a better understanding of some parts of the clone lattice $\mathcal{L}_A$.

The problem of classifying all monoids on a finite set A according to the cardinalities of the corresponding monoidal intervals was first raised by Á. Szendrei [31]. For the case when A is a two-element set Post's description of the clone lattice provides a complete solution to this problem: there are three finite and three infinite intervals. For the case when A is a finite set with more than two elements, and hence the clone lattice has cardinality $2^{\aleph_0}$, I. G. Rosenberg and N. Sauer in [26] observed that each monoidal interval in $\mathcal{L}_A$ either has cardinality $2^{\aleph_0}$ or is countable.

A transformation monoid M on A is called **collapsing** if the monoidal interval $\mathrm{Int}(M)$ has only one element, namely the clone generated by M.

The description of collapsing monoids in general (e.g., in terms of semigroup theoretical concepts) seems hopeless, however, we know that it is algorithmically decidable whether a monoid is collapsing or not (cf. Theorem 2.1). We should note that, so far, this is the only case for which such a decision algorithm exists.

Despite the fact that 'for most M' the monoidal interval $\mathrm{Int}(M)$ is expected to contain uncountably many clones, there are large intervals in the submonoid lattice of the full transformation monoid such that all members of these intervals are collapsing (cf. M. Dormán [5], Proposition 2.4.).

The article has two aims: the first one is to collect all results about monoidal intervals on 3-element sets, while the second one is to start a systematical study of collapsing monoids on 4-element sets.

On a 3-element set, up to permutations of the set, there are 160 transformation monoids; for 115 of these monoids the cardinalities of the corresponding monoidal intervals have been known from previous articles (see Table 1). To these results we can add two new ones in Theorems 4.1 and 4.5. In these theorems we will show that each of the monoidal intervals corresponding to monoids $M_6^{(3)}$ and $M_{10}^{(3)}$ has cardinality $2^{\aleph_0}$, where $M_k^{(3)}$ denotes the monoid in line k of Table 1.

On a 4-element set, due to limitations of space, we study only transformation monoids with cardinalities $\leqslant 10$. The number of such monoids, up to permutations of the set, is 37642. Among these monoids 56 are collapsing. These results were achieved by using 'brute force method' and a CSP based computer program that was written in Java programming language. Two new classes of collapsing monoids are described in Theorems 3.3 and 3.5 that cover 11 among the 56 monoids.

2. Preliminaries

This section is devoted to a survey of the basic concepts and techniques that will be used in the article. Throughout the paper, the set A is assumed to be the finite set $\{0, 1, \ldots, n-1\}$, where $n \geqslant 3$ is a positive integer. For an element $a \in A$ a tuple of any length whose components coincide with a will be denoted by $\widehat{a}$.

The full transformation semigroup, the symmetric group, and the set of unary constant operations on A will be denoted by $T(A)$, $S(A)$, and $C(A)$, respectively. For an arbitrary element a of A we will use the notation c_a for the unary constant operation on A with value a. A transformation t on A will also be written in the from $t(0)t(1)\ldots t(n-1)$ (e.g., 022 will denote the transformation $\{0,1,2\} \to \{0,1,2\},\ 0 \mapsto 0, 1 \mapsto 2, 2 \mapsto 2$).

A set $\mathcal{C}$ of finitary operations on a set A is said to be a **clone** if it contains all the projections and is closed under composition of operations. It is obvious that the set $\mathcal{O}_A$ of all operations and the set $\mathcal{P}_A$ of all projections on A are clones.

Since the intersection of an arbitrary family of clones on A is also a clone, the set of all clones on A constitutes a complete lattice with respect to the set-theoretic inclusion. This lattice will be denoted by $\mathcal{L}_A$. The greatest and the least elements of $\mathcal{L}_A$ are $\mathcal{O}_A$ and $\mathcal{P}_A$, respectively. Furthermore, we can define the **clone generated by a subset** F of $\mathcal{O}_A$ as the intersection of all clones that contain F. This is the least clone containing F, which will be denoted by $\langle F \rangle$. If F is a finite subset of $\mathcal{O}_A$, say $F = \{f_1, \ldots, f_s\}$, then we write $\langle f_1, \ldots, f_s \rangle$ instead of $\langle \{f_1, \ldots, f_s\} \rangle$. For a positive integer ℓ, the set of all ℓ-ary operations in a clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(\ell)}$.

Let f be an ℓ-ary operation in $\mathcal{O}_A$ ($\ell \in \mathbb{N}$). The operation f **depends on its i^{th} variable** if there are elements $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_\ell \in A$ such that the unary operation

$$A \to A,\ a \mapsto f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_\ell)$$

is not a unary constant operation. We call the operation f **essentially k-ary** ($k \in \mathbb{N},\ k \geqslant 2$) if it depends on exactly k of its variables. If f depends on at most one of its variables, we call f **essentially unary**.

For a natural number k a **k-ary relation on** A is a subset of A^k. A relation is finitary if it is k-ary for a positive integer k. We will denote by $\mathfrak{R}_A$ the set of all finitary relations on A.

Let ℓ and m be positive integers, and let $\rho \in \mathfrak{R}_A$ be an m-ary relation and $f \in \mathcal{O}_A$ be an ℓ-ary operation. We call an $\ell \times m$ matrix $X = (x_{i,j})$ over A a **ρ-matrix** if every row of X belongs to ρ, i.e., $(x_{i,1}, \ldots, x_{i,m}) \in \rho$ for all i ($1 \leqslant i \leqslant \ell$). The operation f is said to **preserve** the relation ρ if for every ρ-matrix $X = (x_{i,j}) \in A^{\ell \times m}$ the m-tuple

$$f(X) \stackrel{\text{def.}}{=} \big(f(x_{1,1}, \ldots, x_{\ell,1}), \ldots, f(x_{1,m}, \ldots, x_{\ell,m})\big)$$

also belongs to ρ. It is obvious that the operation f preserves the relation ρ if and only if ρ is a subalgebra of the algebra $(A; f)^m$.

For a subset R of $\mathfrak{R}_A$ the set of all finitary operations on A that preserve each member of R will be denoted by $\mathrm{Pol}(R)$. If R is finite, say $R = \{\rho_1, \ldots, \rho_s\}$, then we simply write $\mathrm{Pol}(\rho_1,\dots,\rho_s)$. On the other hand, for a subset F of $\mathcal{O}_A$ the set of all finitary relations on A that are preserved by each member of F will be denoted by $\mathrm{Inv}(F)$. If F is finite, say $F=\{f_1,\dots,f_s\}$, then we simply write $\mathrm{Inv}(f_1,\dots,f_s)$.

For every set A the maps

$$\begin{aligned}\mathrm{Inv}\colon P(\mathcal{O}_A)\to P(\mathfrak{R}_A),\ F\mapsto \mathrm{Inv}(F),\\ \mathrm{Pol}\colon P(\mathfrak{R}_A)\to P(\mathcal{O}_A),\ R\mapsto \mathrm{Pol}(R)\end{aligned}$$

define a Galois connection between sets of operations and sets of relations, which is the main tool in our investigation.

To give a more detailed introduction into the concept of a monoidal interval let M be a transformation monoid on A, and let $\mathrm{Int}(M)$ denote the collection of all clones $\mathcal{C}$ on A such that the set of unary operations of $\mathcal{C}$ is M. The clone $\langle M\rangle$ of essentially unary operations generated by M is a member of $\mathrm{Int}(M)$, in fact, it is the least member of $\mathrm{Int}(M)$, so $\mathrm{Int}(M)$ is non-empty. Furthermore, it is clear that every clone $\mathcal{C}$ in $\mathrm{Int}(M)$ is contained in the set

$$\begin{aligned}\mathrm{Sta}(M)=\big\{f(x_1,\dots,x_\ell)\in\mathcal{O}_A\mid \ell\in\mathbb{N},\ \text{and}\\ f(m_1(x),\dots,m_\ell(x))\in M \text{ for all } m_1,\dots,m_\ell\in M\big\},\quad (1)\end{aligned}$$

which is called the **stabilizer** of the monoid M. It is easy to verify that $\mathrm{Sta}(M)$ is a clone on A, therefore $\mathrm{Sta}(M)$ is the largest member of $\mathrm{Int}(M)$. Moreover, we see that a clone $\mathcal{C}\in\mathcal{L}_A$ belongs to $\mathrm{Int}(M)$ if and only if $\langle M\rangle\subseteq\mathcal{C}\subseteq\mathrm{Sta}(M)$. Thus $\mathrm{Int}(M)$ is the interval $[\langle M\rangle,\mathrm{Sta}(M)]$ in the clone lattice $\mathcal{L}_A$. Such an interval is called a **monoidal interval**.

The transformation monoids M and M' on the set A are said to be **conjugate** if there is a permutation $\pi\in S(A)$ such that

$$M'=\big\{\pi^{-1}m\pi\mid m\in M\big\}.$$

It is easy to see that conjugacy of monoids is an equivalence relation on $T(A)$. Moreover, if the transformation monoids M and M' are conjugate then the corresponding monoidal intervals $\mathrm{Int}(M)$ and $\mathrm{Int}(M')$ are isomorphic, hence their cardinalities coincide. We note that isomorphism of transformation monoids does not imply conjugacy of these monoids or equality of cardinalities of the corresponding monoidal intervals; transformation monoids $M^{(3)}_{23}$ and $M^{(3)}_{25}$ in Table 1 provide an example.

Recall from the introduction that if a monoidal interval $\mathrm{Int}(M)$ has only one element, then the transformation monoid M is called **collapsing**. In this case the only element of $\mathrm{Int}(M)$ is $\langle M\rangle$. By a result of J.-U. Grabowski [10] the following statement is true.

Theorem 2.1. *A transformation monoid on a finite set is collapsing if and only if the stabilizer of the monoid contains no essentially binary operations.*

To prove that for a transformation monoid M the monoidal interval $\mathrm{Int}(M)$ has cardinality $2^{\aleph_0}$ the following method of J. Demetrovics and L. Hannák in [4] will be useful.

Let I be a set and $\mathfrak{C} = \{\mathcal{C}_i : i \in I\}$ is a set of clones on A. The set $\mathfrak{C}$ is said to be **independent** if for all $i \in I$ we have that

$$\mathcal{C}_i \not\subseteq \left\langle \bigcup \{\mathcal{C}_j : j \in I \setminus \{i\}\} \right\rangle.$$

An easy consequence of independence of clones is the following: if $\mathfrak{C}$ is a complete join-subsemilattice of $\mathcal{L}_A$ that contains an infinite independent subset then $\mathfrak{C}$ has cardinality $2^{\aleph_0}$ (cf. [4], Proposition 1). We remark that a monoidal interval is an example for a complete join-subsemilattice of $\mathcal{L}_A$.

Let $\mathfrak{R} = \{\rho_i : i \in I\}$ be a set of finitary relations on A. The set $\mathfrak{C}$ is **separated by** $\mathfrak{R}$ if for all $i, j \in I$ we have that $\mathcal{C}_i \subseteq \mathrm{Pol}(\rho_j)$ if and only if $i \neq j$. The significance of separation is that independence is a consequence of it, that is, if $\mathfrak{C}$ is separated by $\mathfrak{R}$ then $\mathfrak{C}$ is independent.

Theorem 2.2 (cf. [4], Proposition 3.)**.** *Let $\mathfrak{C} = \{\mathcal{C}_i : i \in \mathbb{N}\}$ be a set of clones separated by a set of relations $\mathfrak{R} = \{\rho_i : i \in \mathbb{N}\}$ on A. Let $\mathcal{K}_1 \subseteq \mathcal{K}_2$ be clones on A such that $\mathcal{C}_i \subseteq \mathcal{K}_2$ and $\mathcal{K}_1 \subseteq \mathrm{Pol}(\rho_i)$ hold for all i ($i \in \mathbb{N}$). Then the interval $[\mathcal{K}_1, \mathcal{K}_2] = \{\mathcal{C} \in \mathcal{L}_A : \mathcal{K}_1 \subseteq \mathcal{C} \subseteq \mathcal{K}_2\}$ has cardinality $2^{\aleph_0}$.*

The case when each member of $\mathfrak{C}$ is generated by a single element of $\mathcal{O}_A$, say $\mathcal{C}_i = \langle f_i \rangle$ for all $i \in I$, is especially important for the construction of monoidal intervals of cardinality $2^{\aleph_0}$. The following corollary of Theorem 2.2 will handle this case.

Corollary 2.3. *Let M be a transformation monoid on A, let $\mathfrak{C} = \{\langle f_i \rangle : i \in \mathbb{N}\}$ be a set of subclones of $\mathrm{Sta}(M)$ and let $\mathfrak{R} = \{\rho_i : i \in \mathbb{N}\}$ be a set of relations on A. If $\mathfrak{C}$ is separated by $\mathfrak{R}$ and $M \subseteq \mathrm{Pol}(\rho_i)$ holds for all i ($i \in \mathbb{N}$) then the monoidal interval $\mathrm{Int}(M)$ has cardinality $2^{\aleph_0}$.*

3. Countable monoidal intervals

The set of transformation monoids on a fixed set for which the corresponding monoidal intervals contain countably many clones can be divided into three parts in a natural way:

- collapsing monoids,
- transformation monoids with finite monoidal intervals that are not collapsing,
- transformation monoids with countably infinite monoidal intervals.

The main results of this section are connected with collapsing monoids. We present two new classes of collapsing monoids that will be discussed in Theorems 3.3 and 3.5.

Table 2 contains those transformation monoids on the three-element set $\{0, 1, 2\}$ for which it is known that the corresponding monoidal intervals are finite. However, this table may be uncomplete.

Finally, on three- and four-element sets there is no known transformation monoid with countably infinite monoidal interval.

Proposition 3.1. *Let M be a transformation monoid on A that contains all the unary constant operations and let B be a subset of A such that $2 \leqslant |B| < |A|$. If*

$$m(B) \subseteq B \text{ holds for every transformation } m \in M \setminus C(A), \tag{2}$$

then $f(B \times B) \subseteq B$ holds for every essentially binary operation in the stabilizer of M.

Proof. Let f be an arbitrary binary operation in $\mathrm{Sta}(M)$. Suppose that there are elements $b_1, b_2 \in B$ and $a \in A \setminus B$ such that $f(b_1, b_2) = a$. We will prove that f is the binary constant operation with value a.

First, we remark that transformation $t = f(c_{b_1}, \mathrm{id}_A)$ is in M and coincides with c_a by (2), since $t(b_2) = f(b_1, b_2) = a$.

Let a_1 and a_2 be arbitrary elements of A, and set $s = f(\mathrm{id}_A, c_{a_2})$. Then $s(b_1) = f(b_1, a_2) = t(a_2) = a$ and (2) imply that $s = c_a$. Hence, $f(a_1, a_2) = s(a_1) = a$. This proves the assertion. □

Let θ be a congruence of the algebra $(A; \mathrm{Sta}(M))$. If f is an ℓ-ary operation in $\mathrm{Sta}(M)$ ($\ell \in \mathbb{N}$) then the operation

$$f/\theta \colon (A/\theta)^\ell \to A/\theta,\ (a_1/\theta, \dots, a_\ell/\theta) \mapsto f(a_1, \dots, a_\ell)/\theta$$

is well-defined. It is easy to see that $M/\theta = \{m/\theta \mid m \in M\}$ is a transformation monoid on A/θ. Moreover, the following statement is true.

Proposition 3.2. *Let M be a transformation monoid on A such that $C(A) \subseteq M$, and let θ be a congruence of the algebra $(A; M)$. Then θ is a congruence of the algebra $(A; \mathrm{Sta}(M))$ as well, and for every operation f in $\mathrm{Sta}(M)$, the operation f/θ belongs to $\mathrm{Sta}(M/\theta)$.*

Proof. To prove the first part of the statement, we note that $C(A) \subseteq M$ ensures that the set of unary polynomial operations of $\mathrm{Sta}(M)$ coincides with M.

To prove the second part, let ℓ be a natural number and let f be an ℓ-ary operation in $\mathrm{Sta}(M)$, furthermore, let $m_1/\theta, \dots, m_\ell/\theta$ be arbitrary elements in M/θ ($m_1, \dots, m_\ell \in M$). Then the unary operation $m = f(m_1, \dots, m_\ell)$ is in $\mathrm{Sta}(M)$. To prove that f/θ is in $\mathrm{Sta}(M/\theta)$ we remark that the sequence of maps $(q_0, q_1, \dots) \colon \mathrm{Sta}(M) \to \mathrm{Sta}(M/\theta)$, where

$$q_i \colon \mathrm{Sta}(M)^{(i)} \to \mathrm{Sta}(M/\theta)^{(i)},\ f \mapsto f/\theta$$

for all $i \in \mathbb{N}_0$, is a homomorphism between the clones $\mathrm{Sta}(M)$ and $\mathrm{Sta}(M/\theta)$ (as multisorted algebras) since the map $\mathrm{Sta}(M) \to \mathrm{Sta}(M/\theta)$, $f \mapsto f/\theta$ preserves composition and projections. Hence, we get that

$$(f/\theta)(m_1/\theta, \dots, m_\ell/\theta) = \big(f(m_1, \dots, m_\ell)/\theta\big) = m/\theta. \tag{3}$$

This concludes the proof of the proposition. □

Let B be the set $A \setminus \{n-1\}$ and define a relation $\epsilon_{M,B}$ on M in the following way: transformations $m, m' \in M$ are $\epsilon_{M,B}$-related if and only if restrictions $m|_B$ and $m'|_B$ coincide. It is obvious that $\epsilon_{M,B}$ is an equivalence relation.

Theorem 3.3. *Let M be a transformation monoid on A with the following properties:*

(i) $C(A) \subseteq M$,
(ii) $m(B) = B$ *holds for every transformation* $m \in M \setminus C(A)$,
(iii) $|c_a/\epsilon_{M,B}| = 1$ $(a \in A)$,
(iv) $|m/\epsilon_{M,B}| \leqslant n-1$ *for all* $m \in M \setminus C(A)$, *and if* $|m/\epsilon_{M,B}| = n-1$ *then* $m/\epsilon_{M,B}$ *contains a permutation,*

and finally,

(v) *the monoid* $(M \setminus \{c_{n-1}\})|_B$ *is collapsing.*

Then M is collapsing.

Proof. To obtain a contradiction, we suppose that M is not collapsing. By Theorem 2.1, we can choose an essentially binary operation, say f, in the stabilizer of M. As M fulfills the requirements of Proposition 3.1, inclusion $f(B \times B) \subseteq B$ holds.

As $f|_B$ is in the stabilizer of $(M \setminus \{c_{n-1}\})|_B$, Assumption (v) implies that it does not depend on both of its variables. We may suppose, without loss of generality, that it does not depend on its second variable. Let s and t be the unary operations $f(\mathrm{id}_A, c_0) \in M$ and $f(c_{n-1}, \mathrm{id}_A) \in M$, respectively. For every $b \in B$ the unary operation $f(c_b, \mathrm{id}_A)$ is $\epsilon_{M,B}$-related to $c_{s(b)}$, hence by (iii), it coincides with $c_{s(b)}$. Since f depends on his second variable, t cannot be a unary constant operation. If s would be a unary constant operation then $f(\mathrm{id}_A, c_a)$ was a unary operation for every $a \in A$, hence operation f would be a binary constant operation. This would contradict the choice of f. Hence, neither t nor s are unary constant operations. For arbitrary element $a \in A$ we get that

$$f(\mathrm{id}_A(x), c_a(x)) = f(x, a) = \begin{cases} s(x) & \text{if } x \in B, \\ t(a) & \text{if } x = n-1. \end{cases}$$

Furthermore, since t is not constant, (ii) implies that

$$|t(A)| \geqslant |t(B)| = |B| = n-1.$$

The transformation t cannot be a permutation because in this case $|s/\epsilon_{M,B}| = n$ would hold since the unary operations $f(\mathrm{id}_A, c_b)$ $(b \in A)$ would be pairwise distinct and $\epsilon_{M,B}$-related to s, which contradicts (iv). If $|t(A)| = n-1$ then $t(A) \subseteq B$ by (ii), and $|s/\epsilon_{M,B}| = n-1$, the latter implies that $s/\epsilon_{M,B}$ contains a permutation ξ, hence,

$$n-1 \in \{m(n-1) \mid m \in s/\epsilon_{M,B}\} = t(A) \subseteq B,$$

since $\xi(n-1) = n-1$, which is a contradiction. The proof of the theorem is complete. □

Remark 3.4. *If $n = 3$ then Theorem 3.3 does not give collapsing monoids, since assumption* (v) *of the theorem cannot hold, due to the fact that on a two-element set there is no collapsing monoid.*

However, if $n = 4$ then Theorem 3.3 gives that the monoids $M_k^{(4)}$ for $k \in \{8, 14, 15, 17, 26, 29, 38, 40, 54\}$ (and their conjugates) are collapsing, where $M_k^{(4)}$ denotes the monoid in line k of Table 4.

Theorem 3.5. *Let A be a disjoint union of the sets B_0 and B_1 with $|B_0|, |B_1| \geqslant 2$. If for every transformation $m \in M$ we have that*

(i) *m is either constant or the identity map on B_i ($i \in \{0, 1\}$), moreover,*
(ii) *if m is constant on both of the sets B_0 and B_1 then m is constant on A,*
(iii) *if m is not constant then $m(B_i) \subseteq B_i$ ($i \in \{0, 1\}$),*

then M is collapsing.

Before we start the proof of the theorem, it is worth noting that M has a simple structure. For $k \in \{0, 1\}$ let $\mathcal{T}_k$ denote the following set transformations:

$$\{m \in T(A) : m \text{ is constant on } B_k \text{ with value in } B_k \text{ and } m \text{ acts identically on } A \setminus B_k\}.$$

Then the assumptions of the theorem yield that M is a subset of $\{\mathrm{id}_A\} \cup C(A) \cup \mathcal{T}_0 \cup \mathcal{T}_1$. Since composition of a transformation from $\mathcal{T}_0$ and a transformation form $\mathcal{T}_1$ is a transformation that is constant on both of the sets B_0 and B_1 with distinct values, monoid M is an arbitrary subset of either $\{\mathrm{id}_A\} \cup C(A) \cup \mathcal{T}_0$ or $\{\mathrm{id}_A\} \cup C(A) \cup \mathcal{T}_1$ that contains $\{\mathrm{id}_A\} \cup C(A)$.

Proof of Theorem 3.5. Let f be a binary operation in $\mathrm{Sta}(M)$. We will prove that f cannot be an essentially binary operation.

It is straightforward to check that $\theta = B_0^2 \cup B_1^2$ is a congruence of the algebra $(A; M)$. Then, by Proposition 3.2, the binary operation f/θ belongs to M/θ. Let $\mathbf{0}$ and $\mathbf{1}$ denote the sets B_0 and B_1, respectively. Then $M/\theta = \{\mathrm{id}_{A/\theta}, c_{\mathbf{0}}, c_{\mathbf{1}}\}$ and using Post's results in [25], we obtain that $\mathrm{Sta}(M/\theta)$ is the clone $\langle c_{\mathbf{0}}, c_{\mathbf{1}}, \wedge, \vee\rangle$, where $\wedge$ and $\vee$ are the lattice operations with respect to the lattice order $\mathbf{0} \leqslant \mathbf{1}$ on $\{\mathbf{0}, \mathbf{1}\}$. Furthermore, f/θ coincides with one of the following binary operations in $\mathrm{Sta}(M/\theta)^{(2)}$:

$$\begin{array}{c|cc} \wedge & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{array}, \quad \begin{array}{c|cc} \vee & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array},$$

$$\begin{array}{c|cc} c_{\mathbf{0}}^{(2)} & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{array}, \quad \begin{array}{c|cc} c_{\mathbf{1}}^{(2)} & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array}, \quad \begin{array}{c|cc} \pi_1 & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array}, \quad \begin{array}{c|cc} \pi_2 & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{array}.$$

To prove that f is essentially unary, we will use the following simple fact about transformations in M that is a consequence of (iii):

$$\text{if } m \in M \text{ and } m/\theta \in C(A/\theta) \text{ then } m \in C(A). \tag{4}$$

If $f/\theta \in \{c_{\mathbf{0}}^{(2)}, c_{\mathbf{1}}^{(2)}, \pi_1, \pi_2\}$ then f/θ does not depend on both of its variables. Without loss of generality, we may assume that f/θ does not depend on its first variable. Let a be an arbitrary element of A. Then by (3) we get that

$$f(\mathrm{id}_A, c_a)/\theta = (f/\theta)(\mathrm{id}_{A/\theta}, c_{a/\theta})$$

is a unary constant operation on A/θ, hence by (4), the unary operation $f(\mathrm{id}_A, c_a)$ is a constant operation, as well. Therefore, operation f does not depend on its first variable.

If $f/\theta = \wedge$ then for every element $a \in B_0$ the unary operations $f(\mathrm{id}_A, c_a)/\theta$ and $f(c_a, \mathrm{id}_A)/\theta$ are constant, hence, $f(\mathrm{id}_A, c_a)$ and $f(c_a, \mathrm{id}_A)$ are constant by (4) with the same value, say with value $a' \in B_0$. Suppose that there is an element b in B_1 such that the restriction of the unary operation $f(c_b, \mathrm{id}_A)$ on B_1 is the identity operation id_{B_1}. Then the unary operation $f(\mathrm{id}_A, c_{b'})$ violates assumption (ii), where $b' \in B_1 \setminus \{b\}$, since $f(\mathrm{id}_A, c_{b'})|_{B_0}$ is constant with value $a' \in B_0$ and $f(\mathrm{id}_A, c_{b'})(b) = f(b, b') = f(c_b, \mathrm{id}_A)(b') = b'$ imply that $f(\mathrm{id}_A, c_{b'})|_{B_1}$ is constant operation with value $b' \in B_1$. Hence, we get a contradiction.

In a similar way, we get that equality $f/\theta = \vee$ leads to a contradiction, as well. We have thereby proved that the operation f must be essentially unary.

This concluses the proof of the theorem. □

Remark 3.6. *Theorem 3.5 yields, when applied to monoids on a 4-element set, that $M_8^{(4)}$ and $M_{16}^{(4)}$ (and their conjugates) are collapsing.*

4. Monoidal intervals with continuum many elements

The aim of this section is to prove that both of the monoidal intervals corresponding to the monoids $M_6^{(3)} = \{000, 002, 012\}$ and $M_{10}^{(3)} = \{000, 012, 022\}$ have cardinalities $2^{\aleph_0}$. The main tool in our proofs is Corollary 2.3.

Theorem 4.1. *The monoidal interval corresponding to the transformation monoid $M_6^{(3)} = \{000, 002, 012\}$ has continuum many elements.*

Proof. Let α_n and β_m be the following relations on $A = \{0, 1, 2\}$ $(m, n \in \mathbb{N},\ m, n \geqslant 3)$:

$$\alpha_n = \{(2, 1, 0, \dots, 0, 0, 0), \dots, (0, 0, 0, \dots, 0, 2, 1), (1, 0, 0, \dots, 0, 0, 2)\} \subseteq A^n,$$
$$\beta_m = \{(1, 2, 0, \dots, 0, 0, 0), \dots, (0, 0, 0, \dots, 0, 1, 2), (2, 0, 0, \dots, 0, 0, 1)\} \subseteq A^m.$$

Define operations f_n and relations ρ_m $(m, n \in \mathbb{Z},\ m, n \geqslant 3)$ on A as follows:

$$f_n\colon A^n \to A,\ f_n(\mathbf{a}) = \begin{cases} 2 & \text{if } \mathbf{a} \in \alpha_n, \\ 0 & \text{otherwise,} \end{cases}$$
$$\rho_m = \big(\{0, 2\}^m \setminus \{\hat{2}\}\big) \cup \beta_m \subseteq A^m.$$

Our aim is to prove that the sets $\{\langle f_n \rangle \mid n \in \mathbb{N},\ n \geqslant 3\}$ and $\{\rho_m \mid m \in \mathbb{N},\ m \geqslant 3\}$ fulfill the requirements of Corollary 2.3.

Claim 4.2. $f_n \in \mathrm{Sta}(M_6^{(3)})$ *for every natural number $n \geqslant 3$.*

Let $t_1, \ldots, t_n$ be arbitrary transformations in $M_6^{(3)}$ and set $t = f_n(t_1, \ldots, t_n)$. Then

$$\begin{aligned} t(0) &= f_n(t_1(0), \ldots, t_n(0)) \in f_n(\{0\}^n) = \{0\}, \\ t(1) &= f_n(t_1(1), \ldots, t_n(1)) \in f_n(\{0,1\}^n) = \{0\}, \\ t(2) &= f_n(t_1(2), \ldots, t_n(2)) \in f_n(\{0,2\}^n) = \{0\} \end{aligned}$$

ensure that $t = c_0 \in M_6^{(3)}$. Therefore, f belongs to $\mathrm{Sta}(M_6^{(3)})$.

Claim 4.3. *$M_6^{(3)} \subseteq \mathrm{Pol}(\rho_m)$ for every natural number $m \geqslant 3$.*

This claim follows from the observation that for every transformation $t \in M_6^{(3)}$ and for every m-tuple $(a_1, \ldots, a_m) \in \rho_m$ we have that

$$(t(a_1), \ldots, t(a_m)) \in \{(a_1, \ldots, a_m)\} \cup \left(\{0,2\}^m \setminus \{\hat{2}\}\right).$$

Finally, we prove that

$$\{\langle f_n \rangle \mid n \in \mathbb{N},\ n \geqslant 3\}$$

is separated by

$$\{\rho_m \mid m \in \mathbb{N},\ m \geqslant 3\}.$$

Claim 4.4. *$f_n \in \mathrm{Pol}(\rho_m)$ if and only if $m \neq n$ ($m, n \in \mathbb{N},\ m, n \geqslant 3$).*

To prove that $f_n \notin \mathrm{Pol}(\rho_n)$ take the following ρ_n-matrix X_n:

$$X_n = \begin{pmatrix} 1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 2 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then $f_n(X_n) = \hat{2} \notin \rho_n$, which proves the assertion.

Assume m and n to be distinct. Since the range of f_n is $\{0,2\}$, it is enough to prove that $f_n(X) \neq \hat{2}$ holds for arbitrary ρ_m-matrix X. To obtain a contradiction, suppose that there is a ρ_m-matrix $X = (x_{i,j}) \in A^{n \times m}$ for which $f_n(X) = \hat{2}$ holds. For every j ($1 \leqslant j \leqslant m$) let $\mathbf{c}_j$ be the n-tuple $(x_{1,j}, \ldots, x_{n,j})$. Then the equalities $f_n(\mathbf{c}_1) = \cdots = f_n(\mathbf{c}_m) = 2$ imply that

$$\mathbf{c}_1, \ldots, \mathbf{c}_m \in \alpha_n, \tag{5}$$

from which it follows that

$$\text{each column of } X \text{ contains exactly one 1 and exactly one 2.} \tag{6}$$

Suppose that equality $\mathbf{c}_j = \mathbf{c}_{j'}$ holds for some distinct indices $j, j' \in \{1, \ldots, m\}$. Then there is an index i for which $x_{i,j} = x_{i,j'} = 1$ hold. However, this is impossible since each row of a ρ_m-matrix contains at most one 1. Hence, the n-tuples $\mathbf{c}_1, \ldots, \mathbf{c}_m$ are pairwise distinct, which implies that $m < n$. For $k \in \{1,2\}$ let H_k be the set of all indexes $l \in \{1, \dots, n\}$ such that the l^{th} row of X contains an entry with k, that is,

$$H_k = \{l \in \mathbb{N} \mid 1 \leqslant l \leqslant n \text{ and } x_{l,j} = k \text{ for some } j \ (1 \leqslant j \leqslant m)\}.$$

Then $|H_1| = |H_2| = m$ holds by (6). Moreover, $H_1 = \{l +_n 1 \mid l \in H_2\}$ also holds by (5) and the definition of α, where $+_n$ is the binary operation on $\{1, \dots, n\}$ defined by the rule

$$a +_n b = \begin{cases} a + b & \text{if } a + b \leqslant n, \\ a + b - n & \text{if } a + b > n. \end{cases}$$

Suppose that equality $H_1 = H_2$ holds. Let l_0 be an arbitrary element of H_1. Then $l_0 \in H_2$, and so, $l_0 +_n 1 \in H_1 = H_2$. By induction, we can easily prove that $l_0 +_n s \in H_1$ holds for every element $s \in \{1, \dots, n\}$, which leads to a contradiction, since the elements $l_0 +_n s$ $(s \in \{1, \dots, n\})$ are pairwise distinct and belong to H_1, hence, $|H_1| \geqslant n > m = |H_2|$. Therefore, $H_1 \neq H_2$ and we can choose an element $l \in H_1 \setminus H_2$. Then the l^{th} row of X contains 1, but it does not contain 2, which contradicts the fact that X is a ρ_m-matrix.

This concludes the proof of the theorem. □

A slight modification of the proof of Theorem 4.1 yields that the monoidal interval $\mathrm{Int}(M_{10}^{(3)})$ is also uncountable.

Theorem 4.5. *The monoidal interval corresponding to the transformation monoid* $M_{10}^{(3)} = \{002, 012, 022\}$ *has continuum many elements.*

Proof. Let the relations α_n, β_m and ρ_m on A $(m, n \in \mathbb{N},\ m, n \geqslant 3)$ be the same as in the proof of Theorem 4.1. Define operations f'_n $(n \in \mathbb{N},\ n \geqslant 3)$ as follows:

$$f'_n \colon A^n \to A,\ f'_n(\mathbf{a}) = \begin{cases} 2 & \text{if } \mathbf{a} \in \alpha_n \cup \{\hat{2}\}, \\ 0 & \text{otherwise.} \end{cases}.$$

Claim 4.6. $f'_n \in \mathrm{Sta}(M_{10}^{(3)})$ *for every natural number* $n \geqslant 3$.

Let $t_1, \dots, t_n$ be arbitrary transformations in $M_{10}^{(3)}$ and set $t = f'_n(t_1, \dots, t_n)$. Then equalities

$$t(0) = f'_n(t_1(0), \dots, t_n(0)) = f'_n(\hat{0}) = 0,$$
$$t(2) = f'_n(t_1(2), \dots, t_n(2)) = f'_n(\hat{2}) = 2$$

ensure that t is in $\mathrm{Sta}(M_{10}^{(3)})$, hence, f'_n is in $\mathrm{Sta}(M_{10}^{(3)})$.

Claim 4.7. $M_{10}^{(3)} \subseteq \mathrm{Pol}(\rho_m)$ *for every natural number* $m \geqslant 3$.

This claim follows from the observation that for every transformation $t \in M_{10}^{(3)}$ and for every m-tuple $(a_1, \dots, a_m) \in \rho_m$ we have that

$$(t(a_1), \dots, t(a_m)) \in \{(a_1, \dots, a_m)\} \cup \left(\{0, 2\}^m \setminus \{\hat{2}\}\right).$$

Finally, we prove that

$$\{\langle f'_n \rangle \mid n \in \mathbb{N},\ n \geqslant 3\}$$

is separated by

$$\{\rho_m \mid m \in \mathbb{N},\ m \geqslant 3\}\,.$$

Claim 4.8. *$f'_n \in \mathrm{Pol}(\rho_m)$ if and only if $m \neq n$ $(m, n \in \mathbb{N},\ m, n \geqslant 3)$.*

For every natural number $n \geqslant 2$ the matrix X_n, defined in the proof of Claim 4.4, shows that $f'_n \notin \mathrm{Pol}(\rho_n)$.

Assume m and n to be distinct. Since the range of f'_n is $\{0, 2\}$, it is enough to prove that $f_n(X) \neq \hat{2}$ holds for any ρ_m-matrix X. To obtain a contradiction, suppose that there is a ρ_m-matrix $X = (x_{i,j}) \in A^{n\times m}$ for which $f'_n(X) = \hat{2}$ holds. For every j $(1 \leqslant j \leqslant m)$ let $\mathbf{c}_j$ be the n-tuple $(x_{1,j}, \ldots, x_{n,j})$. Then the equalities $f'_n(\mathbf{c}_1) = \cdots = f'_n(\mathbf{c}_m) = 2$ imply that

$$\mathbf{c}_1, \ldots, \mathbf{c}_m \in \alpha_n \cup \{\hat{2}\}. \tag{7}$$

If all the n-tuples $\mathbf{c}_1, \ldots, \mathbf{c}_m$ belong to α_n then we are done: we get a contradiction by the proof of Claim 4.4. Hence, there is an index $j \in \{1, \ldots, m\}$ such that $\mathbf{c}_j = \hat{2}$. There cannot be any other index j' for which $\mathbf{c}_{j'} = \hat{2}$ holds, because in this case each row of X would belong to $\{0, 2\}^m \setminus \{\hat{2}\}$. However, this implies that every column of X coincides with $\hat{2}$ by (7), which is impossible since $\hat{2} \notin \rho_m$. Then every column of X except the j^{th} one belongs to α_n, hence, contains exactly one entry 1. Since the rows of X that contain an entry with 1 coincide, we get that there can be at most one column that belongs to α_n, which is a contradiction.

With this the proof of the theorem is completed. □

5. Computational details

In this section we describe briefly the generation of transformation semigroups and monoids and the checking of the collapsing property of a monoid. Afterward, we describe the software environment, giving some sample codes and indicate possible speedups. The reader who is not interested in technical and implementation details can skip Section 5.2.

5.1. Generating Transformation Semigroups and Monoids and Checking the Collapsing Property. For the computation we represented transformation semigroups and monoids by finite lists of lists. One list enumerates the images of the elements from the base set $A = \{0, 1, 2, 3\}$ under the transformation t in a canonical order, i.e.., $L_t = \{t(0), t(1), t(2), t(3)\}$. Let us assume that list T contains all the transformations L_t. For generating all the transformation semigroups of a fixed size k, on the set $|A| = 4$, we checked the standard closure operation on each k-element subset of the T. In addition, we associated with each monoid a canonical index to save memory by storage. To check the **collapsing property** of a monoid, Theorem 2.1 by Grabowski is used. It is clear that this algorithmic characterization of the collapsing property is in some sense optimal. Using this theorem, it is sufficient to generate all possible essential binary operations and check them one by one to solve the decision problem for collapsing property of monoids. We represented a binary operation by giving its Cayley table again as a finite list of lists. Instead of an exhaustive search among the essentially binary operations a constraint satisfaction paradigm can be applied to solve the decision problem more effectively.

5.2. Implementation Details. We did our computation on a standard linux desktop machine with Mathematica [33] and Java. The *Mathematica* programming language and the available *Combinatorica* package made it possible to write a simple prototype code for generating the semigroups, monoids and for checking the collapsing property of monoids. We give two pieces of code and afterward explain the possible speed-ups.

The variable `A4P` contains all the 256 transformations on $\{0,1,2,3\}$. `TrOp` computes the composition of two transformations and `ClSubset` checks closure. Instead of storing the transformation semigroups as lists of lists, we associate a canonical index to each of the subsets of `A4P`. What we get after the first call is the index list of the transformation semigroups with two elements with offset 256.

Code #1:
```
**DEFINITIONS**
A4P=Tuples[Range[4], 4]-1;
TrOp[el1_List, el2_List] :=Map[el2[[el1[[#]]+1]] &, Range[Length[el1]]]
ClSubset[l_List] :=
Module[{L = Tuples[l, 2]}, Complement[Union[Map[TrOp[#] &, L]], l] == {}]
**CALL**
Flatten[Position[ Map[ClSubset[NthSubset[#, A4P]] &,
Range[1 + 256, 256 + Binomial[256, 2]]], True]]
**OUTPUT**
{1,2,3,4,5,8,...,32612,32634}
```

Now we wish to check whether a given monoid M is collapsing or not. The computation is based on Theorem 2.1. To be concrete, let us assume that variable `M44[[317]]` contains the monoid

$$\{\{0,0,0,0\},\{0,1,2,3\},\{1,1,1,1\},\{2,2,2,2\}\}.$$

The `TestOp` auxiliary function tests whether a binary operation is essential and $\mathrm{Sta}(M)$ contains the operation. Here is a brute force code which does the job.

Code #2:
```
**DEFINITIONS**
PropBinSel[tab_] := Length[Union[tab]]>1 ∧ Length[Union[Transpose[tab]]]>1
CollBinOp[M_, op_] := Module[{LP = Tuples[M, 2]},
Complement[ Union[Table[ Map[op[[Sequence@@(1+#)]] &, Transpose[LP[[j]]]],
{j, Length[LP]}]], M] == {}]
TestOp[M_, op_] := PropBinSel[op] ∧ CollBinOp[M, op]
NthBinOp[n_, A_] := Partition[PadLeft[IntegerDigits[n - 1, A], A^2] + 1, A]-1
**CALL**
Count[Map[TestOp[M44[[317]], NthBinOp[#, 4]] &, Range[1, 4^16]], True]
**OUTPUT**
0
```

Since the result is 0, that is, there is no essential binary operation in Sta(`M44[[317]]`), the exhaustive search confirms that the monoid is collapsing, see the first row of Table 4.

Now we turn to the problem of making the computations more effective. First we emphasize the role of the hardware and software architecture for speeding up. Specifically, we discuss a parallelization option, which is available in the recent version of Mathematica and the possibility of compiling codes. Second, we discuss underlying mathematical theories which makes the generation of semigroups and monoids, and deciding the collapsing property fast.

In Mathematica, if one has a simple subalgorithm which is used often, one has the chance to compile the code of the function and run it faster. Such possible algorithms could be `TrOp`, `ClSubset` for instance. Moreover, from Mathematica 7, one can exploit the fact directly, that recent desktop machines have several cores and Intel CPU even more threads available. Typically if one has a list of homogeneous elements and a certain property should be checked or on each element the same operation should be executed, the command `Parallelize` is handy. The speeding up could be 4-12 times even a simple desktop machine having a processor with 8/12 threads [33].

The generation of monoids of size n could be made faster if we store one canonical representative of all the conjugacy classes of monoids of size at most $n-1$. In the prototype Mathematica implementation we stored the monoids with their canonical indices, but it turned out that one needs big resources if all monoids need to be generated and stored. Therefore we also tried a parallelized C program on a computer grid which was able to generate all the monoids.

For checking if a monoid is collapsing, we have to bring binary operations and monoids into connection. Instead of the brute force search described above in the small piece of Code 2, we express the problem as a constraint satisfaction problem (see e.g. [32]), or more specifically as a Boolean satisfaction problem.

Let M be a fixed monoid on the set A, and suppose that we are searching for an essential binary operation f in the stabilizer of M. For each tuple $(a,b,c) \in A^3$ we introduce a Boolean variable x_{abc} which will express the fact that

$$x_{abc} \equiv f(a,b) = c.$$

These variables encode an operation if and only if for each $a, b \in A$ there exists a unique $c \in A$ for which x_{abc} is true. The existence and uniqueness of c can be expressed by the following Boolean formulae

$$\begin{aligned}\text{Exist} &= \bigwedge_{a,b\in A} \bigvee_{c\in A} x_{abc}, \text{ and}\\ \text{Unique} &= \bigwedge_{a,b\in A} \bigwedge_{\substack{c,d\in A\\ c\neq d}} \neg x_{abc} \vee \neg x_{abd}.\end{aligned}$$

We can also express the fact that f is in the stabilizer of M using the Boolean formula

$$\text{Stabil} = \bigwedge_{s,t\in M} \bigvee_{r\in M} \bigwedge_{a\in A} x_{s(a),t(a),r(a)}.$$

The binary operation depends on its first variable, if there exists elements $a, b, c \in A$ such that $f(a, c) \neq f(b, c)$, which can be expressed as

$$\text{Dep1} = \bigvee_{a,b,c,d\in A} x_{acd} \wedge \neg x_{bcd}.$$

Dependency on the second variable can be expressed in a similar way. Now the Boolean formula

$$\text{Exist} \wedge \text{Unique} \wedge \text{Stabil} \wedge \text{Dep1} \wedge \text{Dep2}$$

is satisfiable if and only if there exists an essentially binary operation in $\mathrm{Sta}(M)$.

With this approach, the execution time of a Java program for the decision of collapsing property even for the biggest monoids has been reduced to a few minutes. One of the authors provides a web application of checking collapsing property

`http://www.math.u-szeged.hu/~mmaroti/applets/CollapsingMonoid.html`

which is linked to the monoid lists in the webpage

`http://www.math.u-szeged.hu/~vajda/CMO`.

6. Conclusion and Future Work

We collected all the available results regarding the cardinality of monoidal intervals for $T(\{0, 1, 2\})$ and extended them with some new results. Here all the collapsing monoids are known, however the cardinality of some monoidal intervals are still unknown. We enumerated all transformation subsemigroups and submonoids of the full transformation monoid $T(\{0, 1, 2, 3\})$. We gave the number of conjugacy classes for monoids. We investigated the collapsing property of transformation submonoids of $T(\{0, 1, 2, 3\})$, which could be seen as a direct continuation of the work described in [5]. To save space, we only give the number of monoids, the number of conjugacy classes and the list of collapsing monoids up to size 10 in this article. We did not enumerate all the collapsing monoids, but a more thorough list can be found on the webpage mentioned above.

It turns out that the sharp algorithmic criteria of Grabowski [10] combined with CSP solvers are good enough to decide the collapsing property of any monoids in $T(\{0, 1, 2, 3\})$. It is clear that with the current computational methodology and technology we could enumerate all the collapsing monoids, but it is also clear that a systematic description and characterization has not yet been done with this work. We will continue this research and propose it as a challenge problem for the scientific community, as well.

Open Problem 6.1. *Give the full enumeration and characterization of the collapsing monoids of $T(\{0, 1, 2, 3\})$. Find the sizes of monoidal intervals on three- and the four-element sets.*

Open Problem 6.2. *Let $M \leqslant C(A) \cup S(A)$ be a transformation monoid on a finite set with at least three elements that contains exactly one unary constant operation. Is it true that the monoidal interval corresponding to M has cardinality continuum?*

7. Tables

This section is devoted to tables. It contains four tables and their descriptions. These tables summarize earlier results from the literature and our new results about certain classes of monoidal intervals.

In Table 1 we summarize known results that concerned with monoidal intervals on $\{0,1,2\}$. In the first column we number the monoids for further reference. The monoids are arranged by the following order $\sqsubseteq_A$ on the set of all submonoids of $T(A)$: if M and M' are submonoids of $T(A)$ then $M \sqsubseteq_A M'$ if either $|M| < |M'|$ or $|M| = |M'|$ and $t_1 = t'_1, \ldots, t_{i-1} = t'_{i-1}$, $t_i \sqsubseteq t'_i$, where $M = \{t_1, \ldots, t_s\}$, $M' = \{t'_1, \ldots, t'_s\}$ and the elements of M and M' are enumerated in their 'natural lexicographic order' $\sqsubseteq$. From each conjugacy class the first monoid of this class with respect to $\sqsubseteq_A$ is selected, and this set of representatives constitutes the table. In the second column there are minimal generating systems for the monoids. In third column of Table 1 the expression $[x/yz]$ $(x, y \in \mathbb{N},\ z \in \{a, b\})$ shows the cardinality x of M, the label y (a positive integer) of the isomorphism class of M, and finally, if the isomorphism class of M splits into two or more conjugacy classes, then the label z (a letter a,b,c,...) indicates the conjugacy class of M within its isomorphism class. The fourth column of the table contains the cardinalities of the conjugacy classes. The fifth and the sixth columns give the cardinality of $\mathrm{Int}(M)$ and references, respectively. In the fifth column the symbol '?' indicates that the cardinality of the corresponding monoidal interval is unknown, so far.

In Table 2 we collect the known finite monoidal intervals on $\{0,1,2\}$. This table is just an extract from Table 1 and we do not know whether is it complete or not.

Table 3 shows some basic facts about the number of monoids and collapsing monoids on $\{0,1,2,3\}$, and the number of these monoids up to conjugacy.

Finally, Table 4 lists all collapsing transformation monoids M on $\{0,1,2,3\}$ with $|M| \leqslant 10$. In this table 'CSPA' indicates that, so far, we have only a computer check that ensures the collapsing property of the monoid. (CSPA is an abbreviation for 'CSP Attack'.)

Table 1: Monoidal intervals on $\{0,1,2\}$

n	Generating set for $M_n^{(3)}$	Iso. class	Numb. of conj.	$\lvert\mathrm{Int}(M_n^{(3)})\rvert$	Ref.
1	$\{012\}$	$[1/1]$	1	$2^{\aleph_0}$	[22, 24]
2	$\{000, 012\}$	$[2/1a]$	3	$2^{\aleph_0}$	[12, 18]
3	$\{002, 012\}$	$[2/1b]$	6	$2^{\aleph_0}$	[3]
4	$\{021\}$	$[2/2]$	3	?	
5	$\{001, 012\}$	$[3/1]$	6	$2^{\aleph_0}$	[18]
6	$\{000, 002, 012\}$	$[3/2a]$	6	$2^{\aleph_0}$	Th. 4.1
7	$\{000, 011, 012\}$	$[3/2b]$	6	$2^{\aleph_0}$	[6]
8	$\{000, 021\}$	$[3/3]$	3	$2^{\aleph_0}$	[30]
9	$\{000, 012, 111\}$	$[3/4a]$	3	4	[18]
10	$\{002, 012, 022\}$	$[3/5]$	3	$2^{\aleph_0}$	Th. 4.5
11	$\{002, 012, 112\}$	$[3/4b]$	3	?	

12	$\{012, 220\}$	[3/6]	6	?	
13	$\{120\}$	[3/7]	1	3	[29]
14	$\{001, 002, 012\}$	[4/1]	6	$2^{\aleph_0}$	[18]
15	$\{001, 010, 012\}$	[4/2a]	6	$2^{\aleph_0}$	[18]
16	$\{001, 011, 012\}$	[4/2b]	6	$2^{\aleph_0}$	[18]
17	$\{001, 012, 111\}$	[4/3]	6	$2^{\aleph_0}$	[18]
18	$\{002, 010, 012\}$	[4/4]	3	?	
19	$\{000, 002, 012, 022\}$	[4/5]	6	?	
20	$\{002, 012, 111\}$	[4/6]	6	?	
21	$\{000, 002, 012, 222\}$	[4/7]	6	?	
22	$\{000, 011, 012, 022\}$	[4/8]	3	?	
23	$\{000, 102\}$	[4/9a]	3	1	[9]
24	$\{000, 012, 111, 222\}$	[4/10]	1	1	[23]
25	$\{002, 102\}$	[4/9b]	3	?	
26	$\{001, 002, 010, 012\}$	[5/1]	6	$2^{\aleph_0}$	[18]
27	$\{002, 011, 012\}$	[5/2]	6	$2^{\aleph_0}$	[18]
28	$\{001, 002, 012, 111\}$	[5/3]	6	$2^{\aleph_0}$	[18]
29	$\{001, 010, 011, 012\}$	[5/4]	6	$2^{\aleph_0}$	[18]
30	$\{001, 010, 012, 111\}$	[5/5a]	6	$2^{\aleph_0}$	[18]
31	$\{001, 011, 012, 111\}$	[5/5b]	6	$2^{\aleph_0}$	[18]
32	$\{001, 012, 110\}$	[5/6]	3	$2^{\aleph_0}$	[18, 21]
33	$\{001, 012, 112\}$	[5/7]	6	$2^{\aleph_0}$	[18]
34	$\{001, 012, 222\}$	[5/8]	6	$2^{\aleph_0}$	[18]
35	$\{002, 010, 012, 111\}$	[5/9]	6	?	
36	$\{000, 002, 012, 022, 222\}$	[5/10]	3	?	
37	$\{000, 002, 012, 112\}$	[5/11]	3	?	
38	$\{002, 012, 111, 222\}$	[5/12]	6	6	[8]
39	$\{000, 012, 220\}$	[5/13]	6	?	
40	$\{000, 011, 021\}$	[5/14]	3	?	
41	$\{000, 021, 111\}$	[5/15]	3	1	[23]
42	$\{002, 012, 200\}$	[5/16]	3	?	
43	$\{002, 012, 221\}$	[5/17]	3	?	
44	$\{002, 010, 011, 012\}$	[6/1]	6	$2^{\aleph_0}$	[18]
45	$\{001, 012, 020\}$	[6/2]	3	?	
46	$\{001, 002, 010, 012, 111\}$	[6/3]	6	$2^{\aleph_0}$	[18]
47	$\{001, 012, 022\}$	[6/4]	6	$2^{\aleph_0}$	[18]
48	$\{002, 011, 012, 111\}$	[6/5]	6	$2^{\aleph_0}$	[18]
49	$\{001, 002, 012, 110\}$	[6/6]	6	$2^{\aleph_0}$	[18, 21]
50	$\{001, 002, 012, 112\}$	[6/7]	6	$2^{\aleph_0}$	[18]
51	$\{001, 002, 012, 222\}$	[6/8]	6	$2^{\aleph_0}$	[18]
52	$\{001, 010, 011, 012, 111\}$	[6/9]	6	$2^{\aleph_0}$	[18]
53	$\{001, 010, 012, 110\}$	[6/10]	6	$2^{\aleph_0}$	[18, 21]
54	$\{001, 010, 012, 222\}$	[6/11]	6	$2^{\aleph_0}$	[18]
55	$\{001, 011, 012, 112\}$	[6/12]	6	$2^{\aleph_0}$	[18]
56	$\{001, 011, 012, 222\}$	[6/13]	6	$2^{\aleph_0}$	[18]
57	$\{001, 102\}$	[6/14]	3	$2^{\aleph_0}$	[18, 21]
58	$\{001, 012, 110, 222\}$	[6/15]	3	$2^{\aleph_0}$	[18, 21]
59	$\{001, 012, 112, 222\}$	[6/16]	6	$2^{\aleph_0}$	[18]
60	$\{002, 012, 101\}$	[6/17]	6	?	
61	$\{002, 010, 012, 111, 222\}$	[6/18]	3	?	
62	$\{002, 012, 022, 111\}$	[6/19]	3	?	
63	$\{000, 002, 102\}$	[6/20]	3	?	
64	$\{000, 002, 012, 112, 222\}$	[6/21]	3	?	
65	$\{012, 111, 220\}$	[6/22]	6	?	
66	$\{000, 120\}$	[6/23]	1	1	[23]
67	$\{002, 210\}$	[6/24]	3	?	
68	$\{102, 220\}$	[6/25]	3	?	
69	$\{021, 102\}$	[6/26]	1	1	[23]
70	$\{002, 010, 011, 012, 111\}$	[7/1]	6	$2^{\aleph_0}$	[18]
71	$\{001, 021\}$	[7/2]	3	?	
72	$\{001, 002, 010, 012, 110\}$	[7/3]	6	$2^{\aleph_0}$	[18, 21]
73	$\{001, 002, 010, 012, 222\}$	[7/4]	6	$2^{\aleph_0}$	[18]
74	$\{002, 011, 012, 110\}$	[7/5]	6	$2^{\aleph_0}$	[18, 21]

75	$\{002, 011, 012, 112\}$	[7/6]	6	$2^{\aleph_0}$	[18]
76	$\{002, 011, 012, 222\}$	[7/7]	6	$2^{\aleph_0}$	[18]
77	$\{001, 002, 012, 110, 112\}$	[7/8]	3	$2^{\aleph_0}$	[18, 21]
78	$\{001, 002, 012, 110, 222\}$	[7/9]	6	$2^{\aleph_0}$	[18, 21]
79	$\{001, 002, 012, 112, 222\}$	[7/10]	6	$2^{\aleph_0}$	[18]
80	$\{001, 010, 011, 012, 110\}$	[7/11]	3	$2^{\aleph_0}$	[18, 21]
81	$\{001, 010, 011, 012, 222\}$	[7/12]	6	$2^{\aleph_0}$	[18]
82	$\{001, 012, 101\}$	[7/13]	6	$2^{\aleph_0}$	[18, 21]
83	$\{001, 010, 012, 110, 222\}$	[7/14]	6	$2^{\aleph_0}$	[18, 21]
84	$\{001, 011, 012, 112, 222\}$	[7/15]	6	$2^{\aleph_0}$	[18]
85	$\{001, 102, 222\}$	[7/16]	3	$2^{\aleph_0}$	[18, 21]
86	$\{002, 012, 101, 222\}$	[7/17]	6	?	
87	$\{000, 002, 012, 200\}$	[7/18]	3	?	
88	$\{000, 002, 102, 222\}$	[7/19]	3	?	
89	$\{001, 010, 012, 022\}$	[8/1]	3	?	
90	$\{002, 010, 011, 012, 110\}$	[8/2]	6	$2^{\aleph_0}$	[18, 21]
91	$\{002, 010, 011, 012, 222\}$	[8/3]	6	$2^{\aleph_0}$	[18]
92	$\{001, 012, 020, 111\}$	[8/4]	3	?	
93	$\{001, 002, 012, 101\}$	[8/5]	6	$2^{\aleph_0}$	[18, 21]
94	$\{001, 002, 010, 012, 112\}$	[8/6]	6	$2^{\aleph_0}$	[18, 21]
95	$\{001, 002, 010, 012, 110, 222\}$	[8/7]	6	$2^{\aleph_0}$	[18, 21]
96	$\{001, 012, 022, 111\}$	[8/8]	6	?	
97	$\{002, 012, 100\}$	[8/9]	6	$2^{\aleph_0}$	[18, 21]
98	$\{002, 011, 012, 110, 222\}$	[8/10]	6	$2^{\aleph_0}$	[18, 21]
99	$\{002, 011, 012, 112, 222\}$	[8/11]	6	$2^{\aleph_0}$	[18]
100	$\{001, 002, 102\}$	[8/12]	6	$2^{\aleph_0}$	[18, 21]
101	$\{001, 002, 012, 110, 112, 222\}$	[8/13]	3	$2^{\aleph_0}$	[18, 21]
102	$\{001, 012, 220\}$	[8/14]	6	$2^{\aleph_0}$	[18, 21]
103	$\{001, 010, 011, 012, 110, 222\}$	[8/15]	3	$2^{\aleph_0}$	[18, 21]
104	$\{001, 012, 101, 222\}$	[8/16]	6	$2^{\aleph_0}$	[18, 21]
105	$\{001, 012, 122\}$	[8/17]	3	?	
106	$\{012, 101, 220\}$	[8/18]	3	?	
107	$\{002, 012, 111, 200\}$	[8/19]	3	?	
108	$\{000, 002, 210\}$	[8/20]	3	?	
109	$\{000, 002, 012, 221\}$	[8/21]	3	?	
110	$\{001, 011, 021\}$	[9/1]	3	$2^{\aleph_0}$	[15]
111	$\{002, 010, 011, 012, 112\}$	[9/2]	3	$2^{\aleph_0}$	[18, 21]
112	$\{002, 010, 011, 012, 110, 222\}$	[9/3]	6	$2^{\aleph_0}$	[18, 21]
113	$\{001, 021, 111\}$	[9/4]	3	?	
114	$\{002, 012, 101, 112\}$	[9/5]	6	$2^{\aleph_0}$	[18, 21]
115	$\{001, 002, 012, 101, 222\}$	[9/6]	6	$2^{\aleph_0}$	[18, 21]
116	$\{001, 002, 010, 012, 112, 222\}$	[9/7]	6	$2^{\aleph_0}$	[18, 21]
117	$\{001, 010, 012, 220\}$	[9/8]	6	$2^{\aleph_0}$	[17, 21]
118	$\{002, 012, 100, 222\}$	[9/9]	6	$2^{\aleph_0}$	[18, 21]
119	$\{011, 012, 220\}$	[9/10]	6	$2^{\aleph_0}$	[17, 21]
120	$\{001, 002, 102, 222\}$	[9/11]	3	$2^{\aleph_0}$	[18, 21]
121	$\{001, 010, 012, 100\}$	[9/12]	3	$2^{\aleph_0}$	[18, 21]
122	$\{002, 111, 210\}$	[9/13]	3	?	
123	$\{000, 102, 220\}$	[9/14]	3	?	
124	$\{000, 021, 102\}$	[9/15]	1	2	[23]
125	$\{001, 010, 012, 022, 111\}$	[10/1]	3	?	
126	$\{002, 010, 012, 100\}$	[10/2]	6	$2^{\aleph_0}$	[18, 21]
127	$\{002, 010, 011, 012, 112, 222\}$	[10/3]	3	$2^{\aleph_0}$	[18, 21]
128	$\{010, 011, 012, 220\}$	[10/4]	6	$2^{\aleph_0}$	[17, 21]
129	$\{002, 012, 101, 112, 222\}$	[10/5]	6	$2^{\aleph_0}$	[18, 21]
130	$\{001, 012, 101, 220\}$	[10/6]	6	$2^{\aleph_0}$	[17, 21]
131	$\{001, 002, 012, 122\}$	[10/7]	3	7	[21]
132	$\{012, 100, 220\}$	[10/8]	6	$2^{\aleph_0}$	[17, 21]
133	$\{001, 002, 012, 221\}$	[10/9]	3	$2^{\aleph_0}$	[17, 21]
134	$\{001, 010, 102\}$	[10/10]	3	$2^{\aleph_0}$	[18, 21]
135	$\{001, 010, 012, 100, 222\}$	[10/11]	3	$2^{\aleph_0}$	[18, 21]
136	$\{001, 011, 021, 111\}$	[11/1]	3	$2^{\aleph_0}$	[21]

137	$\{002, 010, 012, 100, 112\}$	[11/2]	3	$2^{\aleph_0}$	[18, 21]
138	$\{002, 010, 012, 100, 222\}$	[11/3]	6	$2^{\aleph_0}$	[18, 21]
139	$\{002, 010, 012, 221\}$	[11/4]	6	$2^{\aleph_0}$	[17, 21]
140	$\{001, 102, 220\}$	[11/5]	3	$2^{\aleph_0}$	[17, 21]
141	$\{001, 010, 102, 222\}$	[11/6]	3	$2^{\aleph_0}$	[18, 21]
142	$\{002, 010, 102\}$	[12/1]	3	$2^{\aleph_0}$	[15, 18, 21]
143	$\{002, 010, 012, 100, 112, 222\}$	[12/3]	3	$2^{\aleph_0}$	[18, 21]
144	$\{010, 012, 100, 220\}$	[12/3]	6	$2^{\aleph_0}$	[17, 21]
145	$\{002, 010, 011, 012, 221\}$	[12/4]	3	$2^{\aleph_0}$	[17, 21]
146	$\{001, 012, 202\}$	[12/5]	3	1	[5]
147	$\{002, 012, 101, 221\}$	[12/6]	6	$2^{\aleph_0}$	[17, 21]
148	$\{001, 012, 200\}$	[12/7]	6	1	[5]
149	$\{002, 010, 102, 222\}$	[13/1]	3	$2^{\aleph_0}$	[18, 21]
150	$\{021, 101\}$	[13/2]	3	1	[5]
151	$\{002, 010, 012, 100, 221\}$	[14/1]	3	$2^{\aleph_0}$	[17, 21]
152	$\{010, 102, 220\}$	[15/1]	3	$2^{\aleph_0}$	[17, 21]
153	$\{001, 010, 012, 200\}$	[16/1]	3	?	
154	$\{001, 002, 012, 121\}$	[16/2]	3	1	[5]
155	$\{001, 021, 100\}$	[17/1]	3	$2^{\aleph_0}$	[15, 21]
156	$\{001, 021, 112\}$	[17/2]	3	1	[5]
157	$\{001, 012, 020, 122\}$	[22/1]	1	3	[28]
158	$\{001, 021, 122\}$	[23/1]	3	3	[28]
159	$\{001, 120\}$	[24/1]	1	3	[28]
160	$\{001, 021, 102\}$	[27/1]	1	4	[2]

Table 2: The known finite monoidal intervals on $\{0, 1, 2\}$

n	Generating set for $M_n^{(3)}$	Iso. class	Numb. of conj.	$\lvert\mathrm{Int}(M_n^{(3)})\rvert$	Ref.
9	$\{000, 012, 111\}$	$[3/4a]$	3	4	[18]
13	$\{120\}$	[3/7]	1	3	[29]
23	$\{000, 102\}$	$[4/9a]$	3	1	[9]
24	$\{000, 012, 111, 222\}$	[4/10]	1	1	[23]
38	$\{002, 012, 111, 222\}$	[5/12]	6	6	[8]
41	$\{000, 021, 111\}$	[5/15]	3	1	[23]
66	$\{000, 120\}$	[6/23]	1	1	[23]
69	$\{021, 102\}$	[6/26]	1	1	[23]
124	$\{000, 021, 102\}$	[9/15]	1	2	[23]
131	$\{001, 002, 012, 122\}$	[10/7]	3	7	[21]
146	$\{001, 012, 202\}$	[12/5]	3	1	[5]
148	$\{001, 012, 200\}$	[12/7]	6	1	[5]
150	$\{021, 101\}$	[13/2]	3	1	[5]
154	$\{001, 002, 012, 121\}$	[16/2]	3	1	[5]
156	$\{001, 021, 112\}$	[17/2]	3	1	[5]
157	$\{001, 012, 020, 122\}$	[22/1]	1	3	[28]
158	$\{001, 021, 122\}$	[23/1]	3	3	[28]
159	$\{001, 120\}$	[24/1]	1	3	[28]
160	$\{001, 021, 102\}$	[27/1]	1	4	[2]

Table 3: Transformation monoids M on $\{0, 1, 2, 3\}$ with $\lvert M\rvert \leqslant 10$

$\lvert M\rvert$	Numb. of monoids	Up to conj.	Numb. of coll. monoids	Up to conj.
1	1	1	0	0
2	49	6	0	0
3	394	27	0	0
4	1805	105	4	1
5	6066	302	13	2
6	18690	880	97	8
7	48536	2177	76	7
8	113107	4975	150	13
9	234261	10128	148	12
10	444008	19041	124	13
Σ	**866917**	**37642**	**612**	**56**

Table 4: Collapsing transformation monoids M on $\{0,1,2,3\}$ with $|M| \leqslant 10$

n	$\lvert M_n^{(4)}\rvert$	Generating set for $M_n^{(4)}$	Numb. of conj.	Ref.
1	4	$\{0000, 0123, 1111, 2222\}$	4	[9, 18]
2	5	$\{0000, 0132, 2222\}$	12	[9]
3	5	$\{0000, 0123, 1111, 2222, 3333\}$	1	[23]
4	6	$\{0011, 0123, 0202\}$	12	CSPA
5	6	$\{0011, 0123, 0220\}$	24	CSPA
6	6	$\{0022, 0123, 0220, 1111\}$	12	CSPA
7	6	$\{0123, 1111, 2203\}$	24	[9]
8	6	$\{0023, 0123, 1111, 2222, 3333\}$	12	Th. 3.3, Th. 3.5
9	6	$\{0000, 0132, 1111, 2222\}$	6	[23]
10	6	$\{0000, 1032, 2222\}$	3	[23]
11	6	$\{0000, 1203\}$	4	[9]
12	7	$\{0011, 0213\}$	12	[6]
13	7	$\{0022, 0123, 0220, 1111, 3333\}$	6	CSPA
14	7	$\{0032, 0123, 1111, 2222\}$	12	Th. 3.3
15	7	$\{0023, 0123, 0223, 1111, 3333\}$	12	Th. 3.3
16	7	$\{0000, 0023, 0123, 1123, 2222, 3333\}$	6	Th. 3.5
17	7	$\{0123, 1111, 2203, 3333\}$	24	Th. 3.3
18	7	$\{0000, 0231, 1111\}$	4	[23]
19	8	$\{0001, 0101, 0123, 0330\}$	24	CSPA
20	8	$\{0011, 0022, 0123, 0303\}$	12	CSPA
21	8	$\{0011, 0123, 0202, 1111\}$	12	CSPA
22	8	$\{0011, 0123, 0220, 1111\}$	24	CSPA
23	8	$\{0011, 0123, 1313\}$	12	CSPA
24	8	$\{0022, 0321, 1111\}$	6	CSPA
25	8	$\{0022, 0123, 1111, 2002\}$	12	CSPA
26	8	$\{0023, 0132, 1111, 2222\}$	12	Th. 3.3
27	8	$\{0000, 0023, 1023, 2222, 3333\}$	6	CSPA
28	8	$\{0000, 0023, 0123, 2213\}$	12	CSPA
29	8	$\{0123, 1111, 2230\}$	12	Th. 3.3
30	8	$\{0000, 0132, 1023, 2222\}$	3	[23]
31	8	$\{0000, 2310\}$	3	[23]
32	9	$\{0011, 0213, 1111\}$	12	CSPA
33	9	$\{0011, 0123, 0202, 3333\}$	12	CSPA
34	9	$\{0011, 0123, 0220, 3333\}$	24	CSPA
35	9	$\{0011, 0123, 1313, 2222\}$	12	CSPA
36	9	$\{0022, 1111, 2103\}$	12	CSPA
37	9	$\{0022, 0123, 1111, 2002, 3333\}$	6	CSPA
38	9	$\{0023, 0123, 0332, 1111\}$	16	Th. 3.3
39	9	$\{0000, 0023, 0123, 1132, 2222\}$	6	CSPA
40	9	$\{0023, 0123, 1111, 2003, 3333\}$	12	Th. 3.3
41	9	$\{0000, 1023, 2203\}$	12	CSPA
42	9	$\{0000, 0023, 0123, 2213, 3333\}$	12	CSPA
43	9	$\{0000, 0132, 2103\}$	4	[23]
44	10	$\{0001, 0101, 0123, 0330, 1111\}$	24	CSPA
45	10	$\{0011, 0101, 0123, 0220, 1111\}$	12	CSPA
46	10	$\{0011, 0213, 3333\}$	12	CSPA
47	10	$\{0011, 0022, 0123, 0220, 1221\}$	12	CSPA
48	10	$\{0321, 1111, 2002\}$	6	CSPA
49	10	$\{0022, 1111, 2103, 3333\}$	6	CSPA
50	10	$\{0022, 0220, 1111, 2301\}$	6	CSPA
51	10	$\{0000, 0023, 0132, 1123, 2222\}$	6	CSPA
52	10	$\{0000, 0032, 1023, 2222\}$	6	CSPA
53	10	$\{0000, 0023, 1032, 2222\}$	6	CSPA
54	10	$\{0023, 1111, 2103, 3333\}$	12	Th. 3.3
55	10	$\{0000, 1023, 2203, 3333\}$	12	CSPA
56	10	$\{0000, 0132, 0213, 1111\}$	4	[23]

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